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ORIGINAL PAPER

Competition in a chemostat with Beddington–DeAngelis growth rates and periodic pulsed nutrient

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Abstract A system of impulsive differential equations is considered as a model of two populations competing for a pulsed inputting nutrient with Beddington–DeAngelis growth rates. Criteria are derived for the coexistence or non-coexistence of the competing species.

Keywords Competition · Beddington–DeAngelis growth rate · Chemostat · Pulsed input

1 Introduction and the model

As well known, countless organisms live in seasonally or diurnally forced environment, in which the populations obtain food, so the effects of this forcing may be quite profound. Recently, many papers studied chemostat model with variations in the supply of nutrients or the washout. Chemostat with periodic inputs are studied in [1–5], those with periodic washout rate in [6,7], and those with periodic input and washout in [8]. A chemostat is a common laboratory apparatus used to culture microorganisms. Sterile growth medium enters the chemostat at a constant rate; the volume within the chemostat is held constant by allowing excess medium (and microbes) to flow out through a siphon. In this paper, we want to study a chemostat with periodically variable pulsed input. We inoculate this chemostat with two bacteria that compete, in the medium, an abundance of all necessary nutrients but one. This last nutrient is the

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limiting substrate; it is pulsed in periodically. The specific growth rates of bacteria are Beddington–DeAngelis type [9]. Without loss of generality, we assume that the input occur variable at k-times' ($k \in N$) in period T. The model takes the form:

$$\begin{cases} \left.\begin{aligned} \frac{dS}{dT} &= -DS - \frac{\mu_1}{\delta_1}\frac{SP_1}{(A_1+S+B_1P_1)} - \frac{\mu}{\delta_2}\frac{SP_2}{(A_2+S+B_2P_2)}, \\ \frac{dH_1}{dT} &= \frac{\mu_1 SP_1}{A_1+S+B_1P_1} - DP_1, \\ \frac{dH_2}{dT} &= \frac{\mu_2 SP_2}{A_2+S+B_2P_2} - DP_2, \end{aligned}\right\} & T \neq \frac{n\tau+\tau_i}{D}, \\ \Delta S(\frac{n\tau}{D}) = p_i S_0, \quad p_i = \tau_i - \tau_{i-1}, & T = \frac{n\tau+\tau_i}{D}, \\ i = 1, 2, \ldots, k; \qquad n \in N. \end{cases} \tag{1.1}$$

where τ is the period of the impulsive effect and $\tau_0 = 0 < \tau_1 < \tau_2 < \cdots < \tau_k = \tau$ are the k-times of the impulsive effects in per period τ. The state variables S, P_1 and P_2 represent the concentration of limiting substrate, two predators. D is the dilution rate; μ_1 and μ_2 are the uptake constances of the two predators; $\delta_i (i = 1, 2)$ are the yield of predator per unit mass of prey; $\frac{S}{A_i+S+B_iP_i}$ $(i = 1, 2)$ are Beddington–DeAngelis growth rates; $\frac{\tau}{D}$ is the period of the pulsing; τS_0 is the amount of limiting substrate pulsed each $\frac{\tau}{D}$. DS_0 units of substrate are added, on average, per unit of time. $n \in N$, N is the set of all non-negative integers.

The theory of impulsive differential equation appears as a natural description of several real processes subject to certain perturbations whose duration is negligible in comparison with the duration of the process. Recently, equations of this kind are found in a almost every domain of applied sciences. Numerous examples are given in Bainov's and his collaborator's books [10,11]. Some impulsive differential equations have been recently introduced in population dynamics in relation to: impulsive birth [12], impulsive vaccination [13,14], chemotherapeutic treatment of disease [15] and population ecology [16,17].

There are advantages in analyzing dimensionless equations. We treat the reciprocal of the dilution rate as natural measure of time:

$$x \equiv \frac{S}{S_0}, \quad y \equiv \frac{P_1}{\delta_1 S_0}, \quad z \equiv \frac{P_2}{\delta_2 S_0}, \quad t \equiv DT.$$

After some algebra, this yields

$$\begin{cases} \left.\begin{aligned} \frac{dx}{dt} &= -x - \frac{m_1xy}{a_1+x+b_1y} - \frac{m_2xz}{a_2+z+b_2z}, \\ \frac{dy}{dt} &= \frac{m_1xy}{a_1+x+b_1y} - y, \\ \frac{dz}{dt} &= \frac{m_2xz}{a_2+x+b_2z} - z, \end{aligned}\right\} & t \neq n\tau+\tau_i, (i=1, 2, \ldots, k) \\ x((n\tau+\tau_i)^+) = x(n\tau+\tau_i) + p_i, & t = n\tau+\tau_i, (i=1, 2, \ldots, k), \end{cases} \tag{1.2}$$

with

$$m_1 = \frac{\mu_1}{D}, \quad a_1 = \frac{A_1}{S_0}, \quad b_1 = B_1\delta_1;$$

$$m_2 = \frac{\mu_2}{D}, \quad a_2 = \frac{A_2}{S_0}, \quad b_2 = B_2\delta_2.$$

The organizations of the paper are as follows. In the next section, we consider the submodel consisting of nutrient and one microbial population. Here, we obtain criteria for extinction of the microbes as well as criteria for the existence and global stability of a positive periodic solution. In Sect. 3, we obtain criteria for the extinction of the second microbial population, as well as criteria for the existence of a positive periodic solution. Furthermore, we discuss the local stability of the positive periodic solution. Based on the local stability, by topological degree theory we show that the considered full chemostat system has a strictly positive periodic solution which is globally asymptotically stable. Finally, a brief discussion on the biological implications and simulating results are contained in Sect. 5.

2 Behavior of a predator subsystem

In the absence of the predator z, system (1.2) reduces to

$$\begin{cases} \left.\begin{aligned} \frac{dx}{dt} &= -x - \frac{m_1 xy}{a_1 + x + b_1 y}, \\ \frac{dy}{dt} &= \frac{m_1 xy}{a_1 + x + b_1 y} - y, \end{aligned}\right\} & t \neq n\tau + \tau_i, (i = 1, 2, \ldots, k) \\ x((n\tau + \tau_i)^+) = x(n\tau + \tau_i) + p_i, & t = n\tau + \tau_i, (i = 1, 2, \ldots, k), \end{cases} \tag{2.1}$$

This nonlinear system has simple periodic solutions. For our purpose, we present these solutions in this sections.

If we add the first and second equations of the system (2.1), we have $\frac{d(x+y)}{dt} = -(x+y)$. If we take variable changes $s = x+y$ then the system (2.1) can be rewritten as

$$\begin{cases} \frac{ds}{dt} = -s, & t \neq n\tau + \tau_i, (i = 1, 2, \ldots, k) \\ s(t^+) = s(t) + p_i, s(0) > 0, & t = n\tau + \tau_i, (i = 1, 2, \ldots, k). \end{cases} \tag{2.2}$$

For the system (2.2), we have the following Lemma 3.1.

Lemma 2.1 *The subsystem (2.2) has a positive periodic solution $\tilde{s}(t)$ and for every solution $s(t)$ of (2.2) we have $|s(t) - \tilde{s}(t)| \to 0$ as $t \to \infty$, where*

$$\begin{cases} \tilde{s}(t) = s_i^+ \exp(-(t - n\tau - \tau_{i-1})), & t \in (n\tau+\tau_{i-1}, n\tau+\tau_i], \\ \tilde{s}(0) = s_0^+ = \dfrac{\sum_{j=1}^k p_j \exp(-\tau + \tau_j)}{1 - \exp(-\tau)}, s_i = s_{i-1}^+ \exp(-p_i). \; i = 1, 2, \ldots, k. \end{cases} \tag{2.3}$$

Proof Suppose $s(t, s_0)$ is a solution of Eq. (2.2), with initial condition $s_0 \in [0, +\infty)$. We have

$$\begin{array}{ll} s(t, s_0) = s((n\tau + \tau_{i-1})^+)\exp(-(t - n\tau - \tau_i), & t \in (n\tau + \tau_{i-1}, n\tau + \tau_i], \\ s(t^+) = s(t) + p_i, & t = n\tau + \tau_i, \end{array} \tag{2.4}$$

for $i = 1, 2, \ldots, k$. We introduce a function $U(s_0) = s(t, y_0)$. For (2.4), we have the following properties:

(i) $0 < s(t, s_0) < \infty$, $t \in (0, \infty)$ is piecewise continuous function;
(ii) The function $U(s_0) = s(t, s_0)$, $s_0 \in (0, \infty)$ is a increasing function.

By direct calculating, we know that the solution $\tilde{s}(t)$ in (2.3) is a τ-period solution of Eq. (2.2); according to (ii), we can see that the solution $\tilde{s}(t)$ is a unique period solution of (2.2). The multiplier μ_s of $\tilde{s}(t)$ is

$$\mu_s := \exp(-\tau) < 1,$$

we can see that $\tilde{s}(t)$ $(t \in (0, \infty))$ is globally asymptotically stable. We complete the proof. □

By the Lemma 2.1, the following lemma is obvious.

Lemma 2.2 *Let* $(x(t), y(t))$ *be any solution of system* (2.1) *with initial condition* $x(0) \geq 0$, $y(0) > 0$, *then* $\lim_{t\to\infty} |x(t) + y(t) - \tilde{s}(t)| = 0$.

The Lemma 2.2 *says that the periodic solution* $\tilde{s}(t)$ *is uniquely invariant manifold of the system* (2.1).

Theorem 2.1 *For the system* (2.1)*, we denote*

$$m_1^* := \tau\left(\int_0^\tau \frac{\tilde{s}(l)}{a_1 + \tilde{s}(l)} dl\right)^{-1}.$$

(1) *If* $m_1 < m_1^*$*, then the system* (2.1) *has a unique globally asymptotically stable positive* $\tau-$*periodic solution* $(x_e(t), y_e(t))$*, where*

$$x_e(t) = 1,\ y_e(t) = 0;$$

(2) *If* $m_1 > m_1^*$*, then the system* (2.1) *has a unique globally asymptotically stable positive* τ*-periodic solution* $(x_s(t), y_s(t))$ *and the* τ*-periodic solution* $(x_e(t), y_e(t))$ *is unstable. The* τ*-period positive solution* $y_s(t)$ *satisfies*

$$\frac{1}{\tau}\int_0^\tau \frac{m_1(\tilde{s}(l) - y_s(l))}{a_1 + \tilde{s}(l) - y_s(l) + b_1 y_s(l)} dl = 1.$$

Proof By Lemma 2.1, we can consider the system (2.1) in its stable invariant manifold $\tilde{s}(t)$, that is

$$\begin{aligned}\frac{dy}{dt} &= \frac{m_1(\tilde{s}(t)-y)y}{a_1+\tilde{s}(t)-y+b_1 y} - y,\\ 0 &\le y_0 \le \tilde{s}(0).\end{aligned} \tag{2.5}$$

Now we prove periodic impulsive Eq. (2.5) has globally stable periodic solution $y_s(t)$. We have the following properties:

(1) $y(t) = y(t, y_0), \quad t \in [0, \infty)$ is continuous function;
(2) $y(t) = y(t, 0) = 0, \quad t \in [0, \infty)$ is a solution;
(3) $y(t) = y(t, \tilde{s}(0)) = \tilde{s}(t), \quad t \in [0, \tau_1]$.

Suppose $y(t, y_0)$ is a solution of Eq. (2.5), with initial condition $y_0 \in [0, \tilde{s}(0)]$. We have

$$\begin{aligned}F(y(t, y_0)) &= \int_0^t \frac{m_1(\tilde{s}(l)-y(l, y_0))}{a_1+\tilde{s}(l)-y(l, y_0)+b_1 y(l, y_0)}dl - t,\\ y(n\tau) &= y_0, \quad t \in (n\tau, (n+1)\tau].\end{aligned} \tag{2.6}$$

For (2.6), we have the following properties:

(i) The function $G(y_0) = y(t, y_0), \quad y_0 \in (0, \tilde{s}(0)]$ is a increasing function;
(ii) $0 < y(t, y_0) < \tilde{s}(t), \quad t \in (0, \infty)$ is continuous function;
(iii) $y(t, 0) = 0, \quad t \in (0, \infty)$ is a solution .

The periodic solutions of (2.5) satisfy the following equation

$$y_0 = y_0 \exp\left(\int_0^\tau \left(\frac{m_1(\tilde{s}(l)-y(l, y_0))}{a_1+(\tilde{s}(l)-y(l, y_0))+b_1 y(l, y_0)} - 1\right) dl\right). \tag{2.7}$$

By (i), (ii) and (iii), we know that if $\frac{1}{\tau}\int_0^\tau \frac{m_1\tilde{s}(l)}{a_1+\tilde{s}(l)}dl > 1$, the Eq. (2.6) has a unique solution in $(0, \tilde{s}(0)]$; otherwise, it has no solution in $(0, \tilde{s}(0)]$.

If $m_1 < m_1^*$, it is obvious that

$$y(t) \le y(0) \exp\left(\left(\frac{1}{\tau}\int_0^\tau \frac{m_1\tilde{s}(l)}{a_1+\tilde{s}(l)}dl - 1\right) t\right) \exp\left(\int_0^t p_1(l)dl\right). \tag{2.8}$$

where $p_1(t) = \frac{m_1\tilde{s}(t)}{a_1+\tilde{s}(t)} - \frac{1}{\tau}\int_0^\tau \frac{m_1\tilde{s}(l)}{a_1+\tilde{s}(l)}dl$; note that $\frac{1}{\tau}\int_0^\tau p_1(l)dl = 0$ and hence that $p_1(t)$ is τ-periodic piecewise continuous function. By $\frac{1}{\tau}\int_0^\tau \frac{m_1\tilde{s}(l)}{a_1+\tilde{s}(l)}dl - 1 < 0$, we obtain that $y(t)$ tends exponentially to zero as $t \to +\infty$. Considering system (2.2), we have $x(t) = s(t) - y(t)$. By Lemma 2.2, we have $\lim_{t\to\infty} |x(t) - \tilde{s}(t)| = 0$. If $m_1 < m_1^*$, then the Eq. (2.5) has stable periodic solution $y_e(t) = 0$. By Lemma 2.2, we have $\lim_{t\to\infty} |x(t) - \tilde{s}(t)| = 0$. We have proved in (1).

If $m_1 > m_1^*$, then Eq. (2.5) has uniquely positive periodic solution. We denote this positive periodic solution

$$y_s(t) = y(t, y_0^*), \qquad x_s(t) = \tilde{s}(t) - y(t, y_0^*),$$

which satisfies the following equation

$$\int_0^\tau \frac{m_1(\tilde{s}(l) - y_s(l))dl}{a_1 + (\tilde{s}(l) - y_s(l)) + b_1 y(l, y_0)} = \tau. \tag{2.9}$$

We denote $y_0^* := y_s(0)$.

For proving the period solution $y_s(t)$, we define a function $F(y(t, y_0)) : (t, y_0) \to R, \in [0, \infty) \times [0, \tilde{s}(0)]$ as following:

$$F(y(t, y_0)) = \int_0^t \frac{m_1(\tilde{s}(l) - y(l, y_0))}{a_1 + (\tilde{s}(l) - y(l, y_0)) + b_1 y(l, y_0)} dl - t.$$

Noticing Eq. (2.5), we have

$$F(y(\tau, y_0)) = \ln\left(\frac{y(\tau, y_0)}{y_0}\right), \quad y_0 \in (0, \tilde{s}(0)]. \tag{2.10}$$

It is obvious that $F(y(n\tau, y_0^*))) = 0$.

For any $y_0 \in (0, \tilde{s}(0))$, by the theorem 2.10 [7] on the differentiability of the solutions on the initial values, $\frac{\partial y(t,y_0)}{\partial y_0}$ exists. Furthermore, $\frac{\partial y(t,y_0)}{\partial y_0} \geq 0$, $t \in (0, \infty)$ is hold (otherwise, there exist $t_0 > 0$, $0 < y_1 < y_2 < \tilde{s}(0)$ such that $y(t_0, y_1) = y(t_0, y_2)$, that is a contradiction with the different flows of system (2.5) not to intersect). And we can have $\tilde{s}(l) > y(l, y_0))$, for $l \in [0, \tau]$. So we obtain that

$$\frac{d(F(y(\tau, y_0)))}{dy_0} < 0. \tag{2.11}$$

So $F(y(\tau, y_0))$, $y_0 \in [0, \tilde{s}(0)]$ is monotonously decreasing continuous function.

Now we set $0 < \varepsilon < y_0^* < \tilde{s}(0)$. According to (2.11), we have that

$$\begin{aligned} &\ln y(\tau, y_0) - \ln y_0 < 0, \quad if \quad y_0^* < y_0 < \tilde{s}(0), \\ &\ln y(\tau, y_0) - \ln y_0 = 0, \quad if \quad y_0 = y_0^*, \\ &\ln y(\tau, y_0) - \ln y_0 > 0, \quad if \quad \varepsilon < y_0 < y_0^*. \end{aligned} \tag{2.12}$$

Furthermore, we obtain the following equations

$$\begin{aligned} &y_0 > y(\tau, y_0) > \cdots > y(n\tau, y_0) > y_0^*, \quad if \quad y_0^* < y_0 \leq \tilde{s}(0), \\ &y_0 < y(\tau, y_0) < \cdots < y(n\tau, y_0) < y_0^*, \quad if \quad \varepsilon \leq y_0 < y_0^*. \end{aligned} \tag{2.13}$$

Set $y_0 \in (0, \tilde{s}(0)]$. According to (2), we suppose that

$$\lim_{n\to\infty} y(n\tau, y_0) = a.$$

We shall prove that the solution $y(t, a)$ is τ-periodic. We note that the functions $y_n(t) = y(t + n\tau, y_0)$, due to the τ-periodicity of Eq. (2.5), are also its solutions and $y_n(0) \to a$ as $n \to \infty$. By the continuous dependence of the solutions on the initial values we have that $y(\tau, a) = \lim_{n\to\infty} y_n(\tau) = a$. Hence the solution $y(t, a)$ is τ-periodic. The periodic solution $y(t, y_0^*)$ is unique, so $a = y_0^*$.

Let $\varepsilon > 0$ be given. By the Theorem 2.9 [7] on the continuous dependence of the solutions on the initial values, there exists a $\delta > 0$ such that

$$|y(t, y_0) - y(t, y_0^*)| < \varepsilon,$$

if $|y_0 - y_0^*| < \delta$ and $0 \le t \le \tau$. Choose $n_1 > 0$ so that $|y(n\tau, y_0) - y_0^*| < \delta$ for $n > n_1$. Then $|y(t, y_0) - y(t, y_0^*)| < \varepsilon$ for $t > n\tau$ which proves that

$$\lim_{n\to\infty} |y(t, y_0) - y(t, y_0^*)| = 0, \quad y_0 \in (0, \tilde{s}(0)].$$

For system (2.1), by Lemma 2.2 we obtain that for any solution $(x(t), y(t))$ with initial condition $x(0) \ge 0$, $y(0) > 0$, $|x - x_s| \to 0$, $|y - y_s| \to 0$ as $t \to \infty$.

From the τ-period solution y_s being globally asymptotically stable, we can obtain that the multiplier μ of y_s, which satisfies

$$\mu = \exp\left(-\int_0^\tau \left(\frac{m_1(a_1 + b_1 y_s(l) + b_1 x_s(l)) y_s(l)}{(a_1 + x_s(l) + b_1 y_s(l))^2} dl\right)\right) < 1. \tag{2.14}$$

where we have used (2.7). This conclusion will be used in the Sect. 4. We have proved (2). $\square$

3 Coexistence of two predators

Suppose $(\tilde{s} - y_s, y_s)$ and $(\tilde{s} - z_s, z_s)$ are positive τ-periodic solutions of system (2.1) and the system

$$\begin{cases} \left.\begin{aligned} \frac{dx}{dt} &= -x - \frac{m_2 x z}{a_2 + x + b_2 z}, \\ \frac{dz}{dt} &= \frac{m_2 x z}{a_2 + x + b_2 z} - z, \end{aligned}\right\} & t \ne n\tau + \tau_i, (i = 1, 2, \ldots, k) \\ x((n\tau + \tau_i)^+) = x(n\tau + \tau_i) + p_i, & t = n\tau + \tau_i, (i = 1, 2, \ldots, k), \end{cases} \tag{3.1}$$

respectively, i.e., $m_1 > m_1^*$ and $m_2 > m_2^* := \tau \left(\int_0^\tau \frac{\tilde{s}(l)}{a_2 + \tilde{s}(l)} dl\right)^{-1}$. Then $(\tilde{s} - y_s, y_s, 0)$ and $(\tilde{s} - z_s, 0, z_s)$ are nonnegative τ-periodic solutions of system (1.2). First we discuss the stability of $(\tilde{s} - y_s, y_s, 0)$.

Theorem 3.1 *If* $(\tilde{s}-y_s, y_s)$ *is a positive asymptotically stable* τ*-periodic solution of system* (2.1)*, then* $(\tilde{s}-y_s, y_s, 0)$ *is asymptotically stable provided*

$$\frac{1}{\tau}\int_0^{\tau}\frac{m_2(\tilde{s}-y_s)}{a_2+(\tilde{s}-y_s)}dl < 1$$

and is unstable if

$$\frac{1}{\tau}\int_0^{\tau}\frac{m_2(\tilde{s}-y_s)}{a_2+(\tilde{s}-y_s)}dl > 1. \tag{3.2}$$

Proof The local stability of periodic solution $(\tilde{s}-y_s, y_s, 0)$ may be determined by considering the behavior of small amplitude perturbations of the solution. Define

$$x(t)=u(t)+\tilde{s}(t)-y_s(t), \quad y(t)=v(t)+y_s(t), \quad z(t)=w(t)$$

there may be written

$$\begin{pmatrix} u(t)\\ v(t)\\ w(t)\end{pmatrix}=\Phi_i(t)\begin{pmatrix} u(0)\\ v(0)\\ w(0)\end{pmatrix} \quad \tau_{i-1}<t<\tau_i, (i=1,2,\ldots,k)$$

where $\Phi_i(t)$ satisfies

$$\frac{d\Phi_i}{dt}=\begin{pmatrix} -1-\frac{m_1y_s(a_1+b_1y_s)}{(a_1+\tilde{s}-y_s+b_1y_s)^2} & -\frac{m_1(\tilde{s}-y_s)(a_1+\tilde{s}-y_s)}{(a_1+\tilde{s}-y_s+b_1y_s)^2} & -\frac{m_2(\tilde{s}-y_s)}{a_2+\tilde{s}-y_s}\\ \frac{m_1y_s(a_1+b_1y_s)}{(a_1+\tilde{s}-y_s+b_1y_s)^2} & \frac{m_1(\tilde{s}-y_s)(a_1+\tilde{s}-y_s)}{(a_1+\tilde{s}-y_s+b_1y_s)^2}-1 & 0\\ 0 & 0 & \frac{m_2(\tilde{s}-y_s)}{a_2+\tilde{s}-y_s}-1\end{pmatrix}\Phi_i(t)$$

and $\Phi_i(\tau_{i-1})=I$, the identity matrix. Hence the fundamental solution matrix is

$$\Phi_i(\tau_i)=\begin{pmatrix}\phi_{1i}(\tau_i) & \phi_{2i}(\tau_i) & *\\ \phi_{3i}(\tau_i) & \phi_{4i}(\tau_i) & **\\ 0 & 0 & \exp\left(\int_{\tau_{i-1}}^{\tau_i}\left(\frac{m_2(\tilde{s}-y_s)}{a_2+\tilde{s}-y_s}-1\right)dl\right)\end{pmatrix}.$$

It is no need to give the exact form of $(*)$ and $(**)$ as it is not required in the analysis that follows. The linearization of impulsive system (1.2) become

$$\begin{pmatrix} u(n\tau_i^+)\\ v(n\tau_i^+)\\ w(n\tau_i^+)\end{pmatrix}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix} u(n\tau_i)\\ v(n\tau_i)\\ w(n\tau_i)\end{pmatrix}$$

We denote that

$$M_i = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Phi_i(\tau), \quad (i = 1, 2, \ldots, k).$$

Hence, we obtain the fundamental solution matrix M is

$$M = M_k \cdots M_2 M_1 = \begin{pmatrix} \phi_{11}(\tau) & \phi_{12}(\tau) & * \\ \phi_{21}(\tau) & \phi_{22}(\tau) & ** \\ 0 & 0 & \exp\left(\int_0^\tau \left(\dfrac{m_2(\tilde{s}(l) - y_s(l))}{a_2 + \tilde{s}(l) - y_s(l)} - 1\right) dl\right) \end{pmatrix}$$

The eigenvalues of the matrix M are $\mu_3 = \exp(\int_0^\tau (m_2 y_s(l) - 1) dl)$ and the eigenvalues μ_1, μ_2 of the following matrix

$$\begin{pmatrix} \phi_{11}(\tau) & \phi_{12}(\tau) \\ \phi_{21}(\tau) & \phi_{22}(\tau) \end{pmatrix}.$$

The μ_1, μ_2 are also the multipliers the locally linearizing system of system (2.1) provided with $m > m_1^*$ at the asymptotically stable periodic solution $(\tilde{s} - y_s, y_s)$, according to Theorem 2.1, we have that $\mu_1 < 1$, $\mu_2 = \mu < 1$.

If $\frac{1}{\tau} \int_0^\tau \frac{m_2(\tilde{s}(l) - y_s(l)))}{a_2 + \tilde{s}(l) - y_s(l)} dl < 1$, then the multiplier $\mu_3 = \exp\left(\int_0^\tau \left(\frac{m_2(\tilde{s}(l) - y_s(l)))}{a_2 + \tilde{s}(l) - y_s(l)} - 1\right) dl\right) < 1$, the boundary periodic solution $(\tilde{s} - y_s, y_s, 0)$ of system (1.2) is locally asymptotically stable.

If $\frac{1}{\tau} \int_0^\tau \frac{m_2(\tilde{s}(l) - y_s(l)))}{a_2 + \tilde{s}(l) - y_s(l)} dl > 1$, then the multiplier $\mu_3 = \exp\left(\int_0^\tau \left(\frac{m_2(\tilde{s}(l) - y_s(l)))}{a_2 + \tilde{s}(l) - y_s(l)} - 1\right) dl\right) > 1$, the boundary periodic solution $(\tilde{s} - y_s, y_s(t), 0)$ of the system (1.2) is unstable. We complete the proof. □

Theorem 3.2 *If $(\tilde{s} - z_s, z_s)$ is a positive asymptotically stable τ-periodic solution of system* (2.1)*, then $(\tilde{s} - z_s, y_s, 0)$ is asymptotically stable provided*

$$\frac{1}{\tau} \int_0^\tau \frac{m_1(\tilde{s} - z_s)}{a_1 + \tilde{s} - z_s} dl < 1$$

and is unstable if

$$\frac{1}{\tau} \int_0^\tau \frac{m_1(\tilde{s} - z_s)}{a_1 + \tilde{s} - z_s} dl > 1. \tag{3.3}$$

What we are really interested in is the existence and stability of a strictly positive τ-periodic solution of system (1.2)*. Thus, in the following theorem, we assume inequalities* (3.2) *and* (3.3) *hold. In order to investigate the dynamical behavor of the predator* *of system (1.2), we add the first, second and third equations of it and take variable changes $s = x + y + z$, then we obtain the following system*

$$\begin{cases} \frac{ds}{dt} = -s, & t \neq n\tau + \tau_i, (i = 1, 2, \dots, k) \\ s(t^+) = s(t) + p_i, s(0) > 0, & t = n\tau + \tau_i, (i = 1, 2, \dots, k). \end{cases}$$

By the Lemma 2.1*, the following lemma is obvious.*

Lemma 3.1 *Let $(x(t), y(t), z(t))$ be any solution of system (1.2) with $X(0) > 0$, then*

$$\lim_{t\to\infty} |x(t) + y(t) + z(t) - \tilde{s}(t)| = 0.$$

By Lemma 3.1*, our attention will mostly focus on the related model:*

$$\begin{cases} \dfrac{dy}{dt} = \dfrac{m_1(\tilde{s} - y - z)y}{a_1 + \tilde{s} - y - z + b_1 y} - y, \\ \dfrac{dz}{dt} = \dfrac{m_2(\tilde{s} - y - z)z}{a_2 + \tilde{s} - y - z + b_2 z} - z, \\ y_0 > 0, z_0 > 0, y_0 + z_0 \leq \tilde{s}(0). \end{cases} \tag{3.4}$$

We first prove the following Lemma.

Lemma 3.2 *Consider*

$$(a)\ \dot{u} = f(t, u) \quad \text{and} \quad (b)\ \dot{v} = f(t, v).$$

Let $f(t, u), g(t, v) : R \times R \to R$ be sufficiently smooth so that solutions to initial value problems exist uniquely and are continuable for $t \geq 0$. Suppose $y^*(t)$ and $z^*(t)$ are attracting τ-periodic solutions of (a) and (b), respectively, i.e., any solution $u(t)$ of (a) and $v(t)$ of (b) satisfy

$$\lim_{t\to\infty} |u(t) - u^*(t)| \to 0 \quad \text{and} \quad \lim_{t\to\infty} |v(t) - v^*(t)| \to 0.$$

Then if $f(t, \cdot) > g(t, \cdot)$, it follows that $u^*(t) > v^*(t)$ for all t.

Proof Case 1. If there exists t_0 such that $u^*(t_0) > v^*(t_0)$, then $u^*(t_0) > v^*(t_0)$ for all $t > t_0$. For otherwise let $t_1 = \sup\{t^*|u^*(t) > v^*(t)\}$ for all $t > t_0$ and $t < t^*$. Then $\dot{u}^*(t_1) = \dot{v}^*(t_1)$ and $u^*(t) > v^*(t)$ for all $t \in [t_0, t_1)$. So $\dot{u}^*(t_1) \leq \dot{v}^*(t_1)$. However, $\dot{u}^*(t_1) - \dot{v}^*(t_1) = f(t_1, u^*(t_1)) - g(t_1, v^*(t_1)) > 0$. This is a contradiction. Further, because u^* and v^* are τ-periodic, then $u^*(t) > v^*(t)$ for all t.

Case 2. If there exists t_0 such that $u^*(t_0) = v^*(t_0)$, then by hypothesis there must be $t_1 > t_0$ and near to t_0 such that $u^*(t_1) > v^*(t_1)$. By Case 1, this is a contradiction.

Case 3. Let $u^*(t) < v^*(t)$ for all t. Suppose $v^*(t_0) = \max_{0\leq t\leq\tau} v^*(t)$ and let $u(t)$ be the solution of (a) satisfying $u(t_0) = v^*(t_0)$. Then similarly to the previous case, $u(t) > v^*(t)$ for all $t > t_0$. Denote $d = dist(u^*, v^*) = \inf_{0\leq t\leq\tau} |u^*(t) - v^*(t)| > 0$.

From the assumption $u^*(t) < v^*(t)$, we have $u(t) - u^*(t) > v(t) - v^*(t) \geq d > 0$. This is a contradiction because $u^*(t)$ is the attracting solution of (a) and the Lemma 3.2 is proved. □

Lemma 3.3 *System (3.4) has at least one positive τ-periodic solution provided inequalities (3.2) and (3.3) hold.*

Proof Step 1. Condition (3.2) guarantees that

$$\frac{dy}{dt} = \frac{m_1(\tilde{s} - z_s - y)y}{a_1 + \tilde{s} - z_s - y + b_1 y} - y \ (y_0 \leq \tilde{s}(0) - z_s(0))$$

has a unique positive τ-periodic solution $y_\infty^*(t)$ which is globally attracting. Condition (3.3) guarantees that

$$\frac{dz}{dt} = \frac{m_2(\tilde{s} - y_s - z)z}{a_2 + \tilde{s} - y_s - z + b_2 z} - z \ (z_0 \leq \tilde{s}(0) - y_s(0))$$

has a unique positive τ-periodic solution $z_\infty^*(t)$ which is globally attracting.

By Lemma 3.2, we know that $y_s(t) > y_\infty^*(t)$ and $z_s(t) > z_\infty^*(t)$ for all $t \geq 0$.

Step 2. Condition $\frac{1}{\tau}\int_0^\tau \frac{m_1(\tilde{s}-z_\infty^*)}{a_1+\tilde{s}-z_\infty^*}dl > \frac{1}{\tau}\int_0^\tau \frac{m_1(\tilde{s}-z_s)}{a_1+\tilde{s}-z_s}dl > 1$ guarantees that

$$\frac{dy}{dt} = \frac{m_1(\tilde{s} - z_1^* - y)y}{a_1 + \tilde{s} - z_1^* - y + b_1 y} - y \ (y_0 \leq \tilde{s}(0) - z_1^*(0))$$

has a unique positive τ-periodic solution $y_1^*(t)$ which is globally attracting. By Theorem 3.3, we know that

$$y_s(t) > y_1^*(t) > y_\infty^*(t), \quad \text{for all } t \geq 0.$$

Step 3. Condition $\frac{1}{\tau}\int_0^\tau \frac{m_2(\tilde{s}-y_1^*)}{a_2+\tilde{s}-y_1^*}dl > \frac{1}{\tau}\int_0^\tau \frac{m_2(\tilde{s}-y_s)}{a_2+\tilde{s}-y_s}dl > 1$ guarantees that

$$\frac{dz}{dt} = \frac{m_2(\tilde{s} - y_1^* - z)z}{a_2 + \tilde{s} - y_1^* - z + b_2 z} - z \ (z_0 \leq \tilde{s}(0) - y_1^*(0))$$

has a unique positive τ-periodic solution $z_1^*(t)$ which is globally attracting. By Theorem 3.3, we know that

$$z_s(t) > z_1^*(t) > z_\infty^*(t), \quad \text{for all } t \geq 0.$$

Step 4. Consider

$$\frac{dy}{dt} = \frac{m_1(\tilde{s} - z_1^* - y)y}{a_1 + \tilde{s} - z_1^* - y + b_1 y} - y \ (y_0 \leq \tilde{s}(0) - z_1^*(0)).$$

Similarly, there exists a unique positive globally attracting τ-periodic solution $y_2^*(t)$ satisfying

$$y_s(t) > y_1^*(t) > y_2^*(t) > y_\infty^*(t), \quad \text{for all } t \geq 0.$$

Step 5. Consider

$$\frac{dz}{dt}=\frac{m_2(\tilde{s}-y_1^*-z)z}{a_2+\tilde{s}-y_1^*-z+b_2z}-z\ (z_0\leq\tilde{s}(0)-y_1^*(0))$$

and obtain a similar solution $z_2^*(t)$ satisfying $z_s(t)>z_2^*(t)>z_1^*(t)>z_\infty^*(t)$ for all $t\geq 0$.

Step 6. According to the proofs in steps 1 and 5 above, we may construct similar related equations giving two monotone sequences $\{y_n^*(t)\}$ and $\{z_n^*(t)\}$ which are positive τ-periodic functions satisfying

$$\tilde{s}(t)>y_s(t)>y_1^*(t)>\cdots>y_n^*(t)>y_{n+1}^*(t)>\cdots>y_\infty^*(t)$$

and

$$\tilde{s}(t)>z_s(t)>\cdots>z_{n+1}^*(t)>z_n^*(t)>\cdots>z_1^*(t)>z_\infty^*(t)$$

for all $t\geq 0$.

Step 7. From the previous steps, there exist functions $y^*(t)$ and $z^*(t)$ defined on $[0,\infty)$ such that

$$\lim_{t\to\infty}y_n^*(t)=y^*(t)\quad\text{and}\quad\lim_{t\to\infty}z_n^*(t)=z^*(t)\quad\text{for all } t\geq 0.$$

Furthermore, $y^*(t)$ and $z^*(t)$ are τ-periodic functions because

$$\begin{aligned}y^*(t+\tau)&=\lim_{t\to\infty}y_n^*(t+\tau)=\lim_{t\to\infty}y_n^*(t)=y^*(t),\\ z^*(t+\tau)&=\lim_{t\to\infty}z_n^*(t+\tau)=\lim_{t\to\infty}z_n^*(t)=z^*(t).\end{aligned}\qquad\square$$

By the τ-periodicity in t and the boundedness of the right-hand sides of all the equations constructed above, it follows that the derivatives of the members of the sequences $\{y_n(t)\}$ and $\{z_n(t)\}$ are bounded in $[0,\infty)$; that is,$\{y_n(t)\}$ and $\{z_n(t)\}$ are uniformly bounded and equicontinuous. Then by virtue of Arzela-Ascoli's lemma [18], for any compact subinterval of $[0,\infty)$, there exist subsequences of $\{y_n(t)\}$ and $\{z_n(t)\}$ which converge uniformly to $y^*(t)$ and $z^*(t)$, respectively, on this subinterval. Thus $y^*(t)$ and $z^*(t)$ are continuous. By the monotonicity of these sequences, we see that the convergences given by (4.5) are uniform on any compact subinterval of $[0,\infty)$. Hence, by Dini's theorem [19], $y^*(t)$ and $z^*(t)$ are continuously differentiable and

$$\begin{aligned}\frac{dy^*}{dt}&=\frac{m_1x^*y^*}{a_1+x^*+b_1y^*}-y^*,\\ \frac{dz^*}{dt}&=\frac{m_2x^*z^*}{a_2+x^*+b_2z^*}-z^*.\end{aligned}$$

The variational system about the positive τ-periodic solution (y^*,z^*) of system (3.4) is

$$\frac{d\Phi}{dt}=\begin{pmatrix}\frac{m_1}{a_1+x^*+b_1y^*}\{x^*-y^*-\frac{(b_1-1)x^*y^*}{a_1+x^*+b_1y^*}\} & \frac{m_1u^*(a_1+b_1y^*)}{(a_1+x^*+b_1y^*)^2}\\ \frac{m_2z^*(a_2+b_2z^*)}{(a_2+x^*+b_2z^*)^2} & \frac{m_2}{a_2+x^*+b_2v^*}\left\{x^*-z^*-\frac{(b_2-1)x^*z^*}{a_2+x^*+b_2z^*}\right\}\end{pmatrix}\Phi.$$

Let

$$Q(t)=\begin{pmatrix}y^*(t) & 0\\ 0 & -z^*(t)\end{pmatrix}.$$

The change of variable $\Psi=Q^{-1}\Phi$ gives

$$\frac{d\Psi}{dt}=\begin{pmatrix}-\frac{m_1y^*(a_1+b_1x^*+b_1y^*)}{(a_1+x^*+b_1y^*)^2} & \frac{m_1z^*(a_1+b_1y^*)}{(a_1+x^*+b_1y^*)^2}\\ \frac{m_2y^*(a_2+b_2z^*)}{(a_2+x^*+b_2z^*)^2} & -\frac{m_2z^*(a_2+b_2x^*+b_2z^*)}{(a_2+x^*+b_2z^*)^2}\end{pmatrix}\Psi:=A(t)\Psi. \quad (3.5)$$

We know that the local stability of (y^*, z^*) is the same as the stability of $(0, 0)$ in Eq. (3.6). For any fixed t, the eigenvalues of $A(t)$ satisfy $\lambda_1(t)+\lambda_2(t)<0$ and $\lambda_1(t)\lambda_2(t)>0$, then $\lambda_1(t)$ and $\lambda_2(t)$ are negative or have negative real part. Hence the Floquet characteristic multipliers of Eq. (3.6) have moduli less than 1 under inequalities (3.2) and (3.3).

In following, we are able to show the asymptotic stability of the τ-periodic solution $(x^*, y^*, z^*):=(\tilde{s}-y^*-z^*, y^*, z^*)$ to be global under inequalities (3.2) and (3.3). Lemma 3.1 implies that (x^*, y^*, z^*) is globally stable with respect to system (1.2) provided that (y^*, z^*) is globally stable with respect to system (3.4). In the following, we are going to show that (y^*, z^*) is a globally stable solution of system under the condition that inequalities (3.2) and (3.3)hold.

Denote by $S=(S_1, S_2)$ the τ-periodic $Poincar\acute{e}$ mapping generated by system (3.4). It is well known that S is a compact operator and every τ-periodic solution of system (3.4) corresponds to a fixed point of S. Clearly, $(0, 0)$, $(y_s, 0)$ and $(0, z_s)$ are all of the τ-periodic solutions of system (3.4) on the boundary of R_+^2. We denote by a^* the fixed points of S in $int(R_+^2)$. For simplicity of notation, we further denote the fixed points of S on the boundary R_+^2 by O, y_s and z_s, respectively. The indices of all fixed points of S in the cone R_+^2 are calculated in the following theorem.

Theorem 3.3 *Assume inequalities* (3.2) *and* (3.3) *hold. Then the following are true.*
(i) $index(S)=1$ *where* i$index(S)=deg(I-S, O)$ *which means the Brouwer degree in the cone* R_+^2*;*
(ii) $index(S, O)=0$*;*
(iii) $index(S, y_s)=index(S, z_s)=0$*;*
(iv) $index(S, a^*)=1$.

Proof Clearly, system (3.4) is point dissipative and S is compact. It follows from [20] or [21] that there exists a connected global attractor $\mathcal{A}$ of S in R_+^2. Hence, all fixed points of S in R_+^2 must be contained in $\mathcal{A}$. Without loss of generality, we suppose $\mathcal{A} \subset [0, K] \times [0, K]$ for certain constant $K > 0$. Recall that system (3.4) is of the form

$$\begin{cases} \dfrac{dy}{dt} = \dfrac{m_1(\tilde{s} - y - z)y}{a_1 + \tilde{s} - y - z + b_1 y} - y, \\ \dfrac{dz}{dt} = \dfrac{m_2(\tilde{s} - y - z)z}{a_2 + \tilde{s} - y - z + b_2 z} - z. \end{cases}$$

As a result, we can choose constant K large enough to guarantee that $[0, K] \times [0, K]$ is not only globally attractive but also positively invariant. Clearly, for any constant $\hat{K} > K$, $[0, \hat{K}) \times [0, \hat{K})$ is positively invariant. Let $\Omega = [0, K+1) \times [0, K+1) \subset R_+^2$. Clearly, Ω is open in R_+^2 with relative boundary $\partial\Omega = \{(y, z) \in R_+^2 : |(y, z)|_{sup} = K + 1\}$. By the excision property of topological degree, it follows that

$$deg(I - S, O) = deg(I - S, \Omega, O).$$

Define a homotopy

$$H(t) = I - tS : \Omega \to R^2.$$

Claim $H(t)$ is Ω-admissible for all $t \in [0, 1]$, i.e., $O \overline{\in} (I - tS)(\partial\Omega)$. It suffices to show that for any $(y, z) \in \partial\Omega$, $(I - tS)(y, z) \neq (0, 0)$. Clearly, when $t = 0$, then $I - tS = I$ and $(0, 0) \overline{\in} \partial\Omega$. When $t = 1$, then $I - tS = I - S$, and since we have supposed that all fixed points of S in R_+^2 are contained in $\mathcal{A} \subset [0, K] \times [0, K]$, hence $(0, 0) \overline{\in} (I - S)(\partial\Omega)$. Similarly, for any $t \in (0, 1)$, $(0, 0) \overline{\in} (I - tS)(\partial\Omega)$ since Ω is positively invariant. This completes the proof of our claim.

Thus, by homotopy invariance, we have

$$\begin{aligned} deg(I - S, \Omega, O) &= deg(H(1), \Omega, O) \\ &= deg(H(0), \Omega, O) \\ &= deg(I, \Omega, O) = 1, \end{aligned}$$

where the last identity is due to the normalization property of topological degree. Hence, $deg(I - S, O) = 1$ and $index(S) = 1$.

(ii) We now prove that index $index(S, O) = 0$. By definition, we know that

$$\mathcal{P}_O = \{(y, z) \in R^2 : O + t(y, z) \in R_+^2 \quad \text{for some} \ \ t > 0\} = R_+^2,$$

and

$$\begin{aligned} \mathcal{S}_O &= \{(y, z) \in \overline{\mathcal{P}}_O : -(y, z) \in \overline{\mathcal{P}}_O\} \\ &= \{(y, z) \in R_+^2 : -(y, z) \in R_+^2\} \\ &= \{O\}. \end{aligned}$$

The variational system about (0, 0) of system (3.4) is

$$\frac{d\Phi}{dt} = \begin{pmatrix} \frac{m_1\tilde{s}(t)}{a_1+\tilde{s}(t)} - 1 & 0 \\ 0 & \frac{m_2\tilde{s}(t)}{a_2+\tilde{s}(t)} - 1 \end{pmatrix}\Phi. \tag{3.6}$$

By a standard argument, one can easily verify that

$$\begin{aligned} DS(O)(y,z) = \Bigg(& y \exp\left(\int_0^\tau \left[\frac{m_1\tilde{s}(t)}{a_1+\tilde{s}(t)} - 1\right] dt\right), \\ & z \exp\left(\int_0^\tau \left[\frac{m_2\tilde{s}(t)}{a_2+\tilde{s}(t)} - 1\right] dt\right)\Bigg), \end{aligned} \tag{3.7}$$

where $DS(O)$ is the *Frechét* derivative of S at O. Furthermore, it follows from inequalities (3.2) and (3.3) that

$$\begin{aligned} r_1 &:= \exp\left(\int_0^\tau \left[\frac{m_1\tilde{s}(t)}{a_1+\tilde{s}(t)} - 1\right] dt\right) > 1, \\ r_2 &:= \exp\left(\int_0^\tau \left[\frac{m_2\tilde{s}(t)}{a_2+\tilde{s}(t)} - 1\right] dt\right) > 1. \end{aligned} \tag{3.8}$$

Now one can easily see that $DS(O)$ has property α (see in Appendix A). Indeed, by definition, it suffices to choose certain $(y,z) \in R_+^2 \backslash \{O\}$ such that, for some $t_0 \in (0,1)$. $t_0 DS(O)(y,z) = (y,z)$, i.e., $(t_0 r_1 y, t_0 r_2 z) = (y,z)$. Clearly, we can choose any $(y,0)$ with $y > 0$ and $t_0 = 1/r_1 \in (0,1)$ since $r_1 > 1$. Furthermore, clearly $I - DS(O)$ is invertible and $O(0,0)$ is an isolated fixed point of S. Therefore, it follows from Theorem A(i) (see in Appendix A) that $index(S, O) = 0$.

(iii) We are going to apply Theorem B (see in Appendix B) to show that

$$index(S, y_s) = index(S, z_s) = 0.$$

It suffices to show $index(S, y_s) = 0$. Similarly, one can show $index(S, z_s) = 0$. Replace A_1, A_2 in Theorem B (see in Appendix B) by S_1, S_2, respectively. One can easily verify that all conditions required in Theorem B are satisfied here. Clearly, we have

$$T = \{y \in R_+ : y = S_1(y, 0)\} = \{0, y_s\},$$

where T is defined in Theorem B. Furthermore, it follows from Eqs.(3.8) and (3.9) that $r_2 > 1$ is the only eigenvalue of $DS_2(0,0)$. The variational system about $(y_s, 0)$ of system (3.4) is

$$\frac{d\Phi}{dt} = \begin{pmatrix} \frac{m_1(\tilde{s}(t)-y_s(t)}{a_1+\tilde{s}(t)-y_s(t)+b_1 y_s(t)} - 1 - \frac{m_1(a_1+b_1\tilde{s}(t))y_s(t)}{(a_1+\tilde{s}(t)-y_s(t)+b_1 y_s(t))^2} & 0 \\ 0 & \frac{m_2(\tilde{s}(t)-y_s(t))}{a_2+\tilde{s}(t)-y_s(t)} - 1 \end{pmatrix}\Phi.$$

Clearly, $r_3 := \exp(\int_0^\tau [\frac{m_2(\tilde{s}(t)-y_s(t))}{a_2+\tilde{s}(t)-y_s(t)} - 1]dt)$ is the only eigenvalue of $DS_2(y_s, 0)$ and $r_3 > 1$ because of inequality (3.2). Therefore, it follows from Theorem B(i) that $index(S, y_s) = 0$. Analogously, it follows from inequality (3.3) that $index(S, z_s)$=0.

(iv) For any fixed points $a^* \in int(R_+^2)$ of S, from the previous assumption, it follows that $\rho(DS(a^*))$ under inequalities (3.2) and (3.3). Note that both(3.2) and (3.3) are independent of a^* itself. According to [[22], Lemma 2(c)], $DS(a^*)$ does not have property. Again by Theorem A(ii), one can easily see that $index(S, a^*) = 1$. We complete the proof. □

Theorem 3.4 *Assume inequalities* (3.2) *and* (3.3) *are valid. Then, there exists a strictly positive τ-periodic solution of system* (1.2) *which is globally asymptotically stable.*

Proof Since system (3.4) generates a discrete monotone dynamical system $\{S^m\}_{m=0}^{\infty}$, we need only to prove the uniqueness of a strictly positive τ-periodic solution of system (3.4). It follows from a simple compactness argument that there are at most finitely many fixed points of S in $int(R_+^2)$. Let them be $\{y_i^* : 1 \le i \le l\}$, where $l \in Z$. From Theorem 3.3, we have $index(S, y_i^*) = 1$, $index(S, O) = 0$, $index(S, y_s) = 0$, $index(S, z_s) = 0$ and $index(S) = 1$. Hence, by the additivity of the fixed point index, it follows that

$$1 = index(S) = index(S, O) + index(S, y_s)$$
$$+index(S, z_s) + \sum_{i=1}^{l} index(S, y_i^*) = l.$$

This implies the uniqueness. Using Lemma 3.2, system (1.2) has a strictly positive τ-periodic solution which is globally asymptotically stable. Therefore, the structure of the global attractor of system (1.2) is very simple, namely, a positive τ-periodic solution. The proof is completed. □

4 Discussion

In this paper, we discussed the dynamical behavior for microbial organisms competing in a chemostat with Beddington–DeAngelis growth rates, and with periodic impulsive nutrient input. In its simplest form, the system approximates conditions for microbial organisms growth in lakes, where the limiting nutrients such as silica and phosphate are supplied from streams draining the watershed. As seasons change, stream drainage patterns change causing variations in the supply of nutrients of lakes. We all know that nutrients are inputted into lakes when rain is falling. In fact, raining is not continuous. It occurs seasonally or in regular pulses. Thus, it is natural to describe this case in impulsive differential equations. For the system, we derived criteria for the coexistence or non-coexistence of the competing species.

In the following, we analyze model (1.2) numerically. In system (1.2), set

$$m_1 = 10, \quad a_1 = 0.8, \quad b_1 = 0.1, \quad a_2 = 1.2, \quad b_2 = 0.7, \quad \tau_1 = 0.6, \quad \tau = \tau_2 = 1.$$

We increase m_2 from 11 to 19. The influences of m_2 may be documented by stroboscopically sampling some of the variables over a range of m_1 values. We numerically integrated system (1.2) for 300 pulsing cycles at each value of m_2. For each m_2, we plotted the last 100 measures of the substrate x, prey y and predator z. Since we sampled at the forcing period, periodic solutions of period τ appear as fixed points, periodic solutions of period 2τ appear as two cycles, and so forth. The resulting bifurcation diagrams (Fig. 1) clear show that: with increasing m_2 from 11 to 19, when $m_2 < m_{min} \approx 11.46$, the predator z is extinct and $(\tilde{s} - y_s, y_s, 0)$ is globally asymptotically stable (Fig. 2 $m_2 = 11.2$); when $m_1 > m_{max} \approx 18.21$, the predator z is extinct and $(\tilde{s} - z_s, 0, z_s)$ is globally asymptotically stable; when $m_{min} < m_2 < m_{max}$, the predators y and z will coexist and $(\tilde{s} - y^* - z^*, y^*, z^*)$ is globally asymptotically stable (Fig. 3 $m_2 = 16.8$).

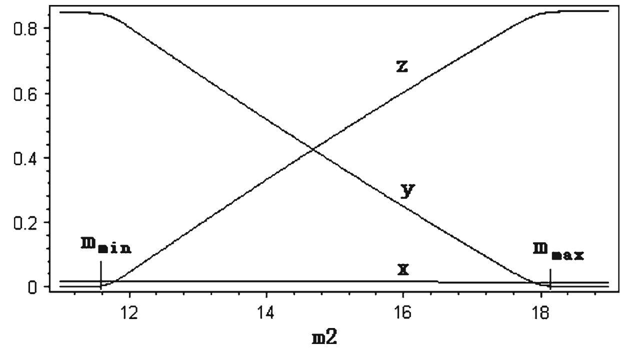

Fig. 1 Bifurcation diagrams of *Poincaré* section for the substrate x, the predators y and z in system (1.2) under $m_1 = 10, a_2 = 1.2, a_1 = 0.8, b_2 = 0.7, b_1 = 0.1, \tau_1 = 0.6, \tau = \tau_2 = 1$ and m_2 is varied in [11,19]

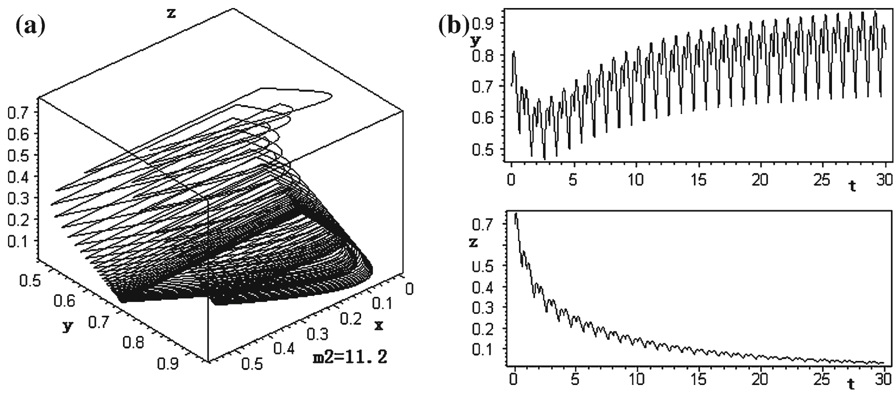

Fig. 2 Extinction of z with $m_2 = 11.2$. (**a**) is the complete trajectories $(x(t), y(t), z(t))$ over the time interval from $t = 1$ to $t = 30$, (**b**) are time series of y and z, respectively

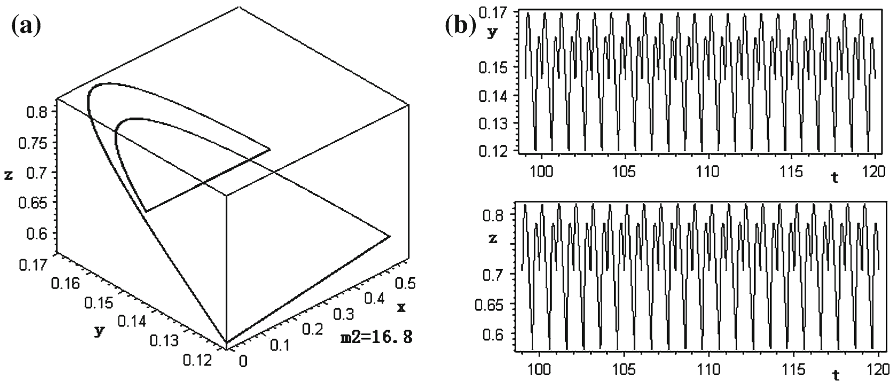

Fig. 3 Coexistence of y and z with $m_2 = 16.8$. (**a**) is the complete trajectories of τ-periodic solution over the time interval from $t = 100$ to $t = 120$, (**b**) are time series of y and z, respectively

Appendix A

Let (E, P) be an ordered Banach space with positive normal cone P. Following [23], for $y \in P$, define

$$\mathcal{P}_z = \{z \in E : z + ty \in P \quad for \quad some \quad t > 0\}.$$

and

$$\mathcal{S}_z = \{y \in \bar{\mathcal{P}}_z : -y \in \bar{\mathcal{P}}_z\}.$$

Let a be a fixed point of some compact operator $T : P \to P$, and denote by $\mathcal{L}$ the $Frech\acute{e}t$ derivative of T at a. We say that $\mathcal{L}$ has property α at a if there exists $t \in (0, 1)$ and $z \in \mathcal{P}_a \backslash \mathcal{S}_a$ such that $z - t\mathcal{L}(z) \in \mathcal{S}_a$. We state a general result of Dancer [23] on fixed point index with respect to the positive cone P (see also [24,25]).

Theorem A (i) *If $I - \mathcal{L}$ is invertible on E, and $\mathcal{L}$ has property α on $\bar{\mathcal{P}}_a$, then $index_P(T, a) = 0$.*

(ii) *If $I - \mathcal{L}$ is invertible on E, and $\mathcal{L}$ does not have property α on $\bar{\mathcal{P}}_a$, then $index_P(T, a) = (-1)^\sigma$, where σ is the sum of the algebraic multiplicities of the eigenvalues of $\mathcal{L}$ whose moduli are greater than* 1.

(iii) *If $I - \mathcal{L}$ is not invertible on E but $Ker(I - \mathcal{L}) \cap \bar{\mathcal{P}}_a = \emptyset$ then $index_P(T, a) = 0$.*

Appendix B

Suppose E_1 and E_2 are ordered Banach spaces with positive cones C_1 and C_2, respectively. Let $E = E_1 \oplus E_2$ and $C = C_1 \oplus C_2$. Then clearly E is an ordered Banach space with positive cone C. Let Ω be an open set in C containing O and $A_i : \overline{\Omega} \to C_i$ be completely continuous operators, (i=1, 2). Denote by (u, v) a general element in C with $u \in C_1$ and $v \in C_2$. Let $A : \overline{\Omega} \to C$ be defined by

$$A(u, v) = (A_1(u, v), A_2(u, v)).$$

Also we define

$$C_2(\varepsilon) = \{v \in C_2 : \|v\|_{E_2} < \varepsilon\}.$$

The following general result of Dancer and Du [[26],Theorem 2.1] on degree calculation is crucial for our applications.

Theorem B *Suppose $U \subset C_1 \cap \Omega$ is relatively open and bounded, and*

$$A_1(u, 0) \neq u \quad \text{for} \quad u \in \partial U,$$
$$A_2(u, 0) \equiv u \quad \text{for} \quad u \in \overline{U}.$$

Suppose $A_2 : \Omega \to C_2$ extends to a continuously differentiable mapping of a neighborhood of Ω into E_2, C_2 is dense in E_2, and $T = \{u \in U : u = A_1(u, 0)\}$. Then the following are true.

(i) $deg_C(I - A, U \times C_2(\varepsilon), 0)$ for $\varepsilon > 0$ small, if for any $u \in T$, the spectral radius $r(A_2\prime(u, 0)|_{C_2}) > 1$ and 1 is not an eigenvalue of $A_2\prime(u, 0)|_{C_2})$ corresponding to a positive eigenvector.

(ii) $deg_C(I - A, U \times C_2(\varepsilon), 0) = deg_{C_1}(I - A_1|_{C_1}, U \times C_2(\varepsilon), 0)$ for $\varepsilon > 0$ small, if for any $u \in T$, the spectral radius $r(A_2\prime(u, 0)|_{C_2}) < 1$.

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